Other derivations of expressions for the radius of the circumscribed sphere were given by Legendre, *Eléments de Géométrie*, 8e éd., Paris, 1809, pp. 302–304 of note V; by Baltzer, *Die Elemente der Mathematik*, Leipzig, volume 2, 1860, pp. 348–349; and by G. Holzmüller, *Elemente der Stereometrie*, Leipzig, volume 2, 1900, pp. 228–231.

ARC.

PROBLEMS—SOLUTIONS.

2801 [1920, 31; 1921, 54–61, 91–97]. Proposed by A. S. HATHAWAY, Houston, Texas.

A dog at the center of a circular pond makes straight for a duck which is swimming along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as $k : 1$, determine the equation of the curve of pursuit and the distance the dog swims to catch the duck.

I. ADDITIONAL REMARKS BY THE PROPOSER.

If we let $d\sigma$ denote differential along the apparent path of the dog, we shall have

$$\left(\frac{d\sigma}{ds}\right)^2 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = MQ^2/a^2,$$

whence $d\sigma/dt = MQv/a$, where $v = ds/dt$ is the real velocity of the duck.

When $k > 1$, $MQ \geqq ka - a$, and so the apparent velocity of the dog within or on the edge of the pond is never less than $(k - 1)v$.

When $k = 1$, the apparent velocity of the dog is never zero within the pond, nor on the edge except at P. In fact the dog will approach P as limiting point.

When $k < 1$, MQ will be zero at the point where the ray $\sin\theta = k$ intersects the circle CKP. Let α be this value of θ and let w be the distance of Q from this point. Then

$$w^2 = r^2 + a^2\cos^2\alpha - 2ar\cos\alpha\cos(\theta - \alpha),$$

and

$$w\,dw/ds = -k(r + a\cos\alpha)[1 - \cos(\theta - \alpha)],$$

which is never positive, and is zero only when $\theta = \alpha$. Therefore w is always decreasing (except when $\theta = \alpha$), and the apparent path of the dog is a spiral converging to the point where the apparent velocity is zero, $\theta = \alpha$, $r = a\cos\alpha$.

It is interesting to note that if the speed of the dog be less than that of the duck, there is one position of starting where the relative positions of the two will remain unchanged, and that this is the limiting goal of the dog from whatever position he starts within the pond.

A number of interesting results may be deduced by determining entrances and exits on fixed curves. Thus on $r = ma\cos\theta$, exits and entrances are divided by the ray $(m - 2)\sin\theta + k = 0$, exits over the shorter arc.

II. REMARK BY H. P. MANNING, Providence, R. I.

Mr. Morley has made a slight mistake (*1921*, 60) in regard to the cusps and inflections of his integral curves. The substitutions on page 55 lead first to an irrational equation for dv/dp, and the second derivative of v for a curve of pursuit is zero only when the integral curve crosses that part of the cubic which lies to the left of the point for which $p = c$. Between the two parts of the cubic all integral curves are concave downwards, but at any point outside of the cubic they curve in opposite directions; and so from a cusp the two branches curve away in opposite directions, one less steep and the other steeper than the slope $-2c$, and the cusps are all ordinary cusps, and not of the rhamphoid type. Indeed, one such cusp is at the point $(0, 1)$, where one branch corresponds to the curve of pursuit and the other to the "curve of flight."

III. REMARKS AND HISTORICAL NOTES BY H. P. MANNING, AND R.C. ARCHIBALD, Brown University.

We have remarked before (*1921*, 92) that the earliest reference then found, to a curve of pursuit where the pursued moved on a circle, was in Ficklin's problem of 1859. Mr. Ball has noted

above that a similar problem, that of a spider and a fly, was proposed, but not solved, in the *Ladies Diary* more than a century earlier. It appears now that this *Diary* problem suggested the following[1]: "To find the nature of the curve, described by a body giving uniform and direct chase to another, supposing the body pursued to move uniformly in the periphery of a given circle." The "solution," published in 1751, leads to Professor Hathaway's equations.

Using Professor Hathaway's notation and figure (*1921*, 94) and regarding the problem as that of a dog and duck, we can describe this "solution" as essentially the following:

First, the two fundamental equations, Professor Hathaway's (3) and (4), are derived.

The velocity with which the dog gains on the duck is equal to his own velocity diminished by the component along QP of the duck's velocity. This gives us at once equation (3).

To get equation (4) take the component along CQ of the dog's velocity. CQ being z, we have

$$dz/ds = k \sin KCQ = k(a \cos \theta - r)/z.$$

But

$$z^2 = r^2 - 2ar \cos \theta + a^2,$$

whence

$$z\, dz = (r - a \cos \theta)dr + ar \sin \theta d\theta,$$

or

$$(a \cos \theta - r)(k + dr/ds) = ar \sin \theta d\theta/ds,$$

which gives at once equation (4).

Ratios are used instead of the trigonometric functions, and x and y are used for r and PK $(= a \cos \theta)$.

The equation in x and y, equivalent to Professor Hathaway's (5), contains the expression $\sqrt{a^2 - y^2}$. It is proposed to develop this in a series and then to use "the method of resolving fluxional equations," a reference being given to Simpson's *Fluxions*. For the particular case of the spider and fly problem it is found that, velocity of the spider : velocity of the fly = 1.16 : 1.

The discussion concludes with the following note: "When the velocity of the body pursued is the greater of the two, the required curve will be a spiral converging continually nearer and nearer to the circumference of a circle concentric with the given one; whose radius is to that of the given one in the ratio of the lesser velocity to the greater."

As a further bibliographical note, reference may be given to problem 2971, proposed by the late Artemas Martin in *Educational Times*, volume 22, 1869, p. 141. It was as follows: "Show that the solution of the famous 'curve of pursuit problem' when the object pursued moves in the circumference of a circle and the pursuer starts from the center, can be made to depend upon the solutions of the differential equations

$$d\theta = -\frac{dt}{n - \cos \phi} \quad (1), \qquad tdt = r\{d(t \sin \phi) - ntd\phi\} \quad (2);$$

where r is the radius of the circle, t the distance the two objects are apart at any time during the motion, ϕ the angle t makes with a tangent to the circle, and θ the arc described by the pursued object from the commencement of the motion, supposing the pursuer to move n times as fast as the pursued." A solution of this problem, by James McMahon, appeared in *Mathematical Questions and Solutions from the "Educational Times"* volume 51, 1889, p. 159.

Captain Henri Brocard, of Bar-le-Duc, France, has kindly drawn our attention to solutions of the following problems (21, 22) proposed by Dr. W. Kapteyn in *Wiskundige Opgaven met de Oplossingen*, new series, volume 10, pp. 50–52, 1907: "21. A point C moves on a circle of radius equal to unity. Another point, situated originally at the center O of the circle, moves with the same velocity as the point C on a curve whose tangent passes constantly through this point. Prove that the radius of curvature at any point M of this curve is equal to the segment [measured from C] intercepted on the radius OC by the normal in M." 22. "Consider the radius of curvature ρ [at M], of the curve referred to in the previous question, as a function of the distance, p, of the origin from the tangent to this curve [at M]. Form the differential equation connecting ρ and p."

Of the first of these problems three solutions were published. In one of them, by the proposer, equations very similar to those employed by Mr. Morley (*1921*, 55) are derived.

The problems were reprinted in *Nouvelles Annales de Mathématiques*, February, 1907, pp. 95, 476; a solution of the first was given on page 173 by M. d'Ocagne, who refers to his paper "Centre de courbure des courbes de poursuite" in *Bulletin de la Société Mathématique de France*, volume 11, 1883, pp. 133–134.

[1] J. Turner, *Mathematical Exercises*, London, 1750–1752; no. 2, 1751, p. 32; no. 3, 1751, pp. 77–80. The problem and "solution" are reprinted in T. Leybourn's *Mathematical Questions proposed in the Ladies' Diary*, vol. 2, 1817, pp. 15–17.